

Quantization with Action-Angle Coherent States

Jean Pierre Gazeau

Astroparticules et Cosmologie, Univ Paris Diderot,

*Sorbonne Paris Cité, 75205 Paris-Fr **

Rina Kanamoto

Division of Advanced Sciences, Ochanomizu University,

2-1-1 Ohtsuka, Bunkyo-ku, Tokyo 112-8610-Jp[†]

Abstract

For a single degree of freedom confined mechanical system with given energy, we know that the motion is always periodic and action-angle variables are convenient choice as conjugate phase-space variables. We construct action-angle coherent states in view to provide a quantization scheme that yields precisely a given observed energy spectrum $\{E_n\}$ for such a system. This construction is based on a Bayesian approach: each family corresponds to a choice of probability distributions such that the classical energy averaged with respect to this probability distribution is precisely E_n up to a constant shift. The formalism is viewed as a natural extension of the Bohr-Sommerfeld rule and an alternative to the canonical quantization. In particular, it also yields a satisfactory angle operator as a bounded self-adjoint operator.

*Electronic address: gazeau@apc.univ-paris7.fr

[†]Electronic address: kanamoto.rina@ocha.ac.jp

I. INTRODUCTION

Action-angle variables

Let us consider a a single degree of freedom confined mechanical system described with phase-space conjugate variables (q, p) . Suppose it conservative. For a given motion its Hamiltonian function is fixed to a certain value E of the energy $H(q, p) = E$. Solving for the momentum variable p leads to $p = p(q, E)$. We suppose that we are in presence of a confinement of the system where we have periodic motions only. Then two types of periodic motions are possible:

- (i) *Libration*: phase trajectory is closed, and then q and p are periodic functions of time with a same period.
- (ii) *Rotation or circulation*: phase trajectory is not closed, then p is periodic function of q .

For either type, introduce the *action variable* [1]

$$J = \oint p(q, E) dq = J(E), \quad (1)$$

where the loop integral is understood as performed over a complete period of libration (resp. rotation). This determines by inversion of $J = J(E)$ the function $E = E(J)$. Now we know that the Hamilton-Jacobi equation $\frac{\partial S}{\partial t} + H(q, \frac{\partial S}{\partial t}) = 0$ obeyed by the action $S = \int L dt$, with $L = p\dot{q} - H$, has a solution of the type $S = W(q, J) - Et$ (note that the action S should not be confused with the action variable J). The time-independent $W = W(J, q) = \int p dq$ is the *Hamilton characteristic function* which generates the contact transformation $(q, p) \mapsto (J, \gamma)$ at constant Hamiltonian, where $\gamma = \frac{\partial W}{\partial J}$ is the angle variable, conjugate to J . It follows from the definition of J that the period τ (resp. frequency ν) of motion at fixed energy is $\tau = \partial J / \partial E = \tau(E)$ (resp. $\nu(E) = 1/\tau(E) = \partial E / \partial J$), and so the time evolution of the angle variable is linear, with period τ :

$$\gamma = \frac{t}{\tau(E)} + \gamma_0 = \nu(E) t + \gamma_0. \quad (2)$$

Note that this equation allows to consider the time as proper to the system at given energy.

Action-angle coherent states for measured energies

Let us suppose that a series of energy measurements on a mechanical system with one-degree of freedom yields an energy spectrum $E_0, E_1, \dots, E_n, \dots$. In this paper families of corresponding

action-angle coherent states are constructed in view to provide a quantization scheme consistent with that discrete sequence of experimental energies. The construction is based on a Bayesian approach: each family corresponds to a choice of probability distributions, $n \mapsto p_n(J)/\mathcal{N}(J)$ (prior discrete), $J \mapsto p_n(J)$ (posterior continuous) such that the mean value of the classical energy with respect to the probability $p_n(J)$ is precisely E_n , up to the addition of a constant independent of n . The formalism [2] can be viewed as a natural extension of the empirical Bohr-Sommerfeld rule, $J(E) = nh$, where h is the Planck constant. We know that this quantization is exact for the motion on the circle (quantization of the angular momentum), and valid in the semiclassical regime. In the deep quantum regime our approach can be viewed as a viable alternative to the canonical quantization, particularly when the latter is impracticable. For instance, it yields a satisfactory angle operator.

In Section II we give an overview of a general construction of coherent states labelled by elements of a measure set, their use as a quantizer frame, and their suitability in regard with their phase space content. We specify this formalism in Section III to the action-angle phase space of a confined system by following a Bayesian construction mentioned above. In Section IV we revisit the question of suitability of these action-angle coherent states. Sections V and VI are devoted to the question of choice of probability distributions appropriate for our quantization goals. Section VII is an illustration of our approach by examining a few characteristics, like time evolution, of two types of coherent states for the free rotator. We end this note in Section VIII with some comments about questions raised by our approach and possible generalizations.

II. COHERENT STATES FAMILY AS HILBERTIAN FRAME FOR PHASE SPACE

A. The general construction [3]

Let X be the phase space of a mechanical system, equipped with its symplectic measure μ . Actually (X, μ) can be any measure space. Let \mathcal{I} be some countable set and \mathcal{O} be an orthonormal system $\mathcal{O} = \{\phi_n, n \in \mathcal{I}\}$ made of elements $\phi_n(x)$ in the Hilbert space $L^2_{\mathbb{C}}(X, \mu)$, with the positiveness and finiteness constraints

$$0 < \mathcal{N}(x) \stackrel{\text{def}}{=} \sum_n |\phi_n(x)|^2 < \infty \quad \text{a.e. } x \in X. \quad (3)$$

Let \mathcal{H} be some separable Hilbert space with orthonormal basis $\{|e_n\rangle, n \in \mathcal{I}\}$ in one-to-one correspondence with elements of \mathcal{O} : $|e_n\rangle \mapsto \phi_n$. Then the following family of states in the companion

\mathcal{H} , labelled by elements of X ,

$$|x\rangle = \frac{1}{\sqrt{\mathcal{N}(x)}} \sum_n \overline{\phi_n(x)} |e_n\rangle, \quad (4)$$

where $\overline{\phi_n(x)}$ refers to the complex conjugate, and obey normalization, $\langle x|x\rangle = 1$, and resolution of the unity $\int_X \mu(dx) \mathcal{N}(x) |x\rangle\langle x| = 1_{\mathcal{H}}$. Due to these two properties, vectors $|x\rangle$ are named coherent states (CS) (in a wide sense). The resolution of the unity **is** precisely the departure point for the corresponding CS quantization of functions (or distributions when the latter are properly defined) on X which transforms them into linear operators in \mathcal{H} :

$$f(x) \mapsto A_f = \int_X \mu(dx) \mathcal{N}(x) f(x) |x\rangle\langle x|. \quad (5)$$

$f(x)$ may be considered as CS quantizable if A_f is densely defined in \mathcal{H} or if the so-called *lower symbol* of A_f , $\check{f}(x) \stackrel{\text{def}}{=} \langle x|A_f|x\rangle$ is a smooth function on X viewed as a topological phase space. Hence, the family $\{|x\rangle\}$, $x \in X$ of coherent states offers a certain point of view or frame (in the right Hilbertian geometrical sense) to analyze, in a non-commutative way, the classical set of points X . Changing the family corresponds to a change of frame and possibly to an equivalent or non equivalent quantization in regard with specific physical quantities, like energy, action, angle, position..., and their mutual Poisson brackets.

B. What we understand by suitable coherent states in Quantum Mechanics

Besides the fundamental quantizer role played by the CS $|x\rangle$, further criteria are usually requested to qualify the latter as suitable from the classical-quantum relation point of view. We list here a few of them. For this purpose, we suppose that the CS $|x\rangle$ depend on a parameter, say h , the Planck constant, such that the classical limit corresponds to $h \rightarrow 0$.

- (i) **Relative error criterion.** For a semi-bounded quantizable function $f(x)$ we define the relative error:

$$\text{rerr}_f(x) = \left| \frac{\check{f}(x) - f(x)}{f(x) + C} \right|, \quad \check{f}(x) \stackrel{\text{def}}{=} \langle x|A_f|x\rangle, \quad (6)$$

where C is a constant which is chosen such that $|f(x) + C| \neq 0$ for all x . Hence: *a CS family is said suitable as a family of quasi-classical states with respect to the function f if $\sup_x \text{rerr}_f(x) \rightarrow 0$ as $h \rightarrow 0$.*

- (ii) **Time evolution criterion in semi-classical regime.** Let $H(x)$ be the Hamiltonian of a mechanical system with phase space X and let A_H be its CS-quantized counterpart. We define the time evolution of the probability density of a CS $|x_0\rangle$ in phase space with respect to the measure $\mu(dx)$ as the function

$$\rho_{x_0}(t, x) \stackrel{\text{def}}{=} \mathcal{N}(x) |\langle x | e^{-iA_H t/\hbar} | x_0 \rangle|^2. \quad (7)$$

Its probabilistic nature is directly derived from the resolution of the unity. Hence: *a CS family is said suitable as a family of quasi-classical states with respect to the time evolution ruled by the quantum Hamiltonian A_H if the support of the function $\rho_{x_0}(t, x)$ tends to locate on the classical phase space trajectory $x(t)$ with initial condition $x(0) = x_0$ as $\hbar \rightarrow 0$.*

- (iii) **Time evolution stability.** *A CS family is said temporally stable if under the action of the evolution operator $e^{-iA_H t/\hbar}$, a coherent state is transformed into another coherent state in the same family, up to possibly a phase factor:*

$$e^{-iA_H t/\hbar} |x\rangle = e^{i\beta(t)} |x(t)\rangle. \quad (8)$$

For more details, see [4] and references therein.

III. A BAYESIAN PROBABILISTIC [5] CONSTRUCTION OF ACTION-ANGLE COHERENT STATES AND RELATED QUANTIZATIONS

Conditional posterior probability distribution

Suppose that measurement on a confined one-dimensional system yields the sequence of values for the energy observable (up to a constant shift):

$$E_0 < E_1 < \dots < E_n < \dots. \quad (9)$$

Supposing a (prior) *uniform* distribution on the range of the action variable J , we define a corresponding sequence of probability distributions $J \mapsto p_n(J)$, i.e. $\int_{\mathbb{R} \text{ or } \mathbb{R}^+} d\tilde{J} p_n(J) = 1$, with $\tilde{J} \stackrel{\text{def}}{=} J/\hbar$, obeying the two conditions:

$$0 < \mathcal{N}(J) \stackrel{\text{def}}{=} \sum_{n \in \mathbb{Z} \text{ or } \mathbb{N}} p_n(J) < \infty, \quad E_n + \text{cst} = \int_{\mathbb{R} \text{ or } \mathbb{R}^+} d\tilde{J} E(J) p_n(J), \quad (10)$$

where \mathbb{R} and \mathbb{Z} (resp. \mathbb{R}^+ and \mathbb{N}) stand for the rotation (resp. libration) type of motion. The finiteness condition allows to consider the map $n \mapsto p_n(J)/\mathcal{N}(J)$ as a probabilistic model referring to the discrete data, which might be viewed in the present context as a *prior distribution* also.

Action-angle coherent states

Let \mathcal{H} be a complex separable Hilbert space with orthonormal basis $\{|e_n\rangle, n \in \mathbb{Z} \text{ or } \mathbb{N}\}$. \mathcal{H} is the space of quantum states. Let $\tau > 0$ be a rescaled period of the angle variable and $X = \{(J, \gamma), J \in \mathbb{R} \text{ or } \mathbb{R}^+, 0 \leq \gamma < \tau\}$ be the action-angle phase space for a rotation (resp. libration) motion with given energies the discrete sequence $E_0 < E_1 < \dots < E_n < \dots$. Let $(p_n(J))_{n \in \mathbb{Z} \text{ or } \mathbb{N}}$ be the sequence of probability distributions associated with these energies. We suppose $p_{-n}(J) = p_n(-J)$ in the rotation case. One then constructs the family of states in \mathcal{H} for the rotation or libration motion as the following continuous map from X into \mathcal{H} :

$$X \ni (J, \gamma) \mapsto |J, \gamma\rangle = \frac{1}{\sqrt{\mathcal{N}(J)}} \sum_n \sqrt{p_n(J)} e^{-i\alpha_n \gamma} |e_n\rangle \in \mathcal{H}, \quad (11)$$

where the choice of the real sequence $n \mapsto \alpha_n$ is left to us in order to comply with some if not all criteria previously listed.

Fundamental properties of action-angle coherent states

In both cases the coherent states $|J, \gamma\rangle$

- (i) are unit vector : $\langle J, \gamma | J, \gamma \rangle = 1$
- (ii) resolve the unity operator in \mathcal{H} with respect a measure “in the Bohr sense” $\mu_B(dJ d\gamma)$ [4] on the phase space X :

$$\int_X \mu_B(dJ d\gamma) \mathcal{N}(J) |J, \gamma\rangle \langle J, \gamma| \stackrel{\text{def}}{=} \int_{-\infty}^{+\infty} d\tilde{J} \mathcal{N}(J) \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} d\gamma |J, \gamma\rangle \langle J, \gamma| = 1_{\mathcal{H}}, \quad (12)$$

Here, we impose $T = 2M\tau$ with $M \in \mathbb{N}$ and letting $M \rightarrow \infty$. Hence, if the sequence (α_n) assumes all its values in $2\pi\mathbb{Z}/\tau$, then the angular integral reduces to $\frac{1}{\tau} \int_0^\tau d\gamma$ and the measure μ_B becomes the ordinary one on the cylinder.

- (iii) allow a “coherent state quantization” of classical observables $f(J, \gamma)$,

$$f(J, \gamma) \mapsto \int_X \mu_B(dJ d\gamma) \mathcal{N}(J) f(J, \gamma) |J, \gamma\rangle \langle J, \gamma| \stackrel{\text{def}}{=} A_f, \quad (13)$$

which is compatible with the energy constraint (10) on the posterior distribution $J \mapsto p_n(J)$. Indeed, the CS quantized version of the classical Hamiltonian $H = E(J)$ is diagonal in

the basis $\{|e_n\rangle, n = 0, 1, \dots\}$ since it is trivially verified that in both cases the quantum Hamiltonian is exactly what we expect:

$$A_{E(J)} = \sum_n (E_n + \text{cst}) |e_n\rangle \langle e_n|. \quad (14)$$

Particular quantizations

Actually, the quantization of any function $f_{\text{act}}(J)$ of the action variable alone yields the diagonal operator:

$$f_{\text{act}}(J) \mapsto A_{f_{\text{act}}} = \sum_n \langle f_{\text{act}} \rangle_n |e_n\rangle \langle e_n|, \quad \langle f_{\text{act}} \rangle_n \stackrel{\text{def}}{=} \int_{\mathbb{R} \text{ or } \mathbb{R}^+} d\tilde{J} f_{\text{act}}(J) p_n(J). \quad (15)$$

On the other hand, the quantization of any τ periodic function $f_{\text{ang}}(\gamma)$ of the angle variable alone yields the operator:

$$f_{\text{ang}}(\gamma) \mapsto A_{f_{\text{ang}}} = \sum_{n, n'} [A_{f_{\text{ang}}}]_{nn'} |e_n\rangle \langle e_{n'}|, \quad (16)$$

where the matrix elements are formally given by:

$$\begin{aligned} [A_{f_{\text{ang}}}]_{nn'} &= \int_{\mathbb{R} \text{ or } \mathbb{R}^+} d\tilde{J} \sqrt{p_n(J) p_{n'}(J)} \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} d\gamma e^{-i(\alpha_n - \alpha_{n'})\gamma} f_{\text{ang}}(\gamma) \\ &= \begin{cases} 0 & \text{if } \alpha_n - \alpha_{n'} \notin \frac{2\pi}{\tau} \mathbb{Z}, \\ \varpi_{nn'} c_k(f_{\text{ang}}; \tau) & \text{if } \alpha_n - \alpha_{n'} = \frac{2\pi}{\tau} k \in \frac{2\pi}{\tau} \mathbb{Z}, \end{cases} \end{aligned} \quad (17)$$

where $c_k(f; \tau) = \frac{1}{\tau} \int_0^\tau d\gamma f(\gamma) e^{-i2\pi k \gamma / \tau}$ is the k th Fourier coefficient of $f(\gamma)$, and $\varpi_{nn'} = \int_{\mathbb{R} \text{ or } \mathbb{R}^+} d\tilde{J} \sqrt{p_n(J) p_{n'}(J)}$ measures correlation between the two distributions $J \mapsto p_n(J)$, $J \mapsto p_{n'}(J)$. Note that the diagonal values are all equal to the average of $f_{\text{ang}}(\gamma)$ over one period. Also the infinite matrix can be sparse, even just diagonal, depending on the choice of the $p_n(J)$ and α_n 's. Hence, the quantization could at end transform classical observables into a commutative algebra of operators.

An important point is that this CS quantization procedure provides, for a given choice of the sequence (α_n) , a self-adjoint angle operator A_γ corresponding to the angle function $\mathcal{A}(\gamma)$ defined on the real line as the τ -periodic extension of $\mathcal{A}(\gamma) = \gamma$ on the interval $[0, \tau)$. Then $c_k(\mathcal{A}; \tau) = i/(2\pi k)$ for $k \neq 0$ and $c_0(\mathcal{A}; \tau) = \tau/2$.

IV. THE QUEST FOR “SUITABLE” COHERENT STATES

From relative error

Since the relative error function involves lower symbols, let us just consider those for the two particular cases of classical functions, $f_{\text{act}}(J)$ and $f_{\text{ang}}(\gamma)$.

$$\begin{aligned}\check{f}_{\text{act}}(J) &= \langle J, \gamma | A_{f_{\text{act}}} | J, \gamma \rangle = \sum_n \langle f_{\text{act}} \rangle_n \frac{p_n(J)}{\mathcal{N}(J)} \equiv \langle \langle f_{\text{act}} \rangle_n \rangle_J, \\ \check{f}_{\text{ang}}(\gamma) &= \langle J, \gamma | A_{f_{\text{ang}}} | J, \gamma \rangle = \sum_{n, n'} \frac{\sqrt{p_n(J) p_{n'}(J)}}{\mathcal{N}(J)} \varpi_{nn'} c_{k(n, n')} e^{i \frac{2\pi}{\tau} k(n, n') \gamma},\end{aligned}$$

where we observe, in the first case, the appearance of a double averaging using the two Bayesian facets, and, in the second case, the presence of a deformation of the Fourier series of $f_{\text{ang}}(\gamma)$ involving, on one hand, the selection rules $k(n, n') \stackrel{\text{def}}{=} \frac{2\pi}{\tau} (\alpha_n - \alpha_{n'}) \in \mathbb{Z}$, and weights of probabilistic origin on the other hand. The relative error (6) expresses relative deviations of the original classical observables to the above types of averaging involving the two facts of the underlying Bayesian duality.

From localization probability distributions

The action-angle phase space representation of a particular coherent state $|J_0, \gamma_0\rangle$, as a function of (J, γ) , is the “normalized” overlap

$$\Psi_{|J_0, \gamma_0\rangle}(J, \gamma) \stackrel{\text{def}}{=} \sqrt{\mathcal{N}(J)} \langle J, \gamma | J_0, \gamma_0 \rangle = \frac{1}{\sqrt{\mathcal{N}(J_0)}} \sum_n \sqrt{p_n(J) p_n(J_0)} e^{i \alpha_n (\gamma - \gamma_0)}, \quad (18)$$

Hence, the map $X \ni (J, \gamma) \mapsto \rho_{|J_0, \gamma_0\rangle}^{\text{phase}}(J, \gamma) \equiv |\Psi_{|J_0, \gamma_0\rangle}(J, \gamma)|^2 = \mathcal{N}(J) |\langle J, \gamma | J_0, \gamma_0 \rangle|^2$ represents a localization probability distribution, namely a generalized version of the Husimi distribution, on the phase space provided with the pseudo-measure μ_B . Indeed, the resolution of the identity gives immediately

$$\int_X \mu_B(dJ d\gamma) \rho_{|J_0, \gamma_0\rangle}^{\text{phase}}(J, \gamma) = 1. \quad (19)$$

If we choose instead a specific realization of the Hilbert space \mathcal{H} , like that one generated by eigenfunctions of the quantum Hamiltonian A_H in “ q ” or “configuration” representation, $|e_n\rangle \mapsto \psi_n(q)$, the corresponding representation of the state $|J_0, \gamma_0\rangle$ reads as

$$\psi_{|J_0, \gamma_0\rangle}(q) = \frac{1}{\sqrt{\mathcal{N}(J_0)}} \sum_n \sqrt{p_n(J_0)} e^{-i \alpha_n \gamma_0} \psi_n(q), \quad (20)$$

with corresponding probability density of localization on the range of the q -variable given by $\rho_{|J_0, \gamma_0\rangle}^{\text{circ}}(q) \equiv |\psi_{|J_0, \gamma_0\rangle}(q)|^2$

From time evolution

Since the CS quantized version A_H of the classical Hamiltonian H is diagonal in the basis $\{|e_n\rangle, n \in \mathbb{Z} \text{ (resp. } \mathbb{N})\}$, the time evolution of the CS in both representations is given respectively by

$$\begin{aligned} e^{-iA_H t/\hbar} \Psi_{|J_0, \gamma_0\rangle}(J, \gamma) &= \sqrt{\mathcal{N}(J)} \langle J, \gamma | e^{-iA_H t/\hbar} | J_0, \gamma_0 \rangle \\ &= \frac{1}{\sqrt{\mathcal{N}(J_0)}} \sum_n \sqrt{p_n(J_0) p_n(J)} e^{i(\alpha_n(\gamma - \gamma_0) - E_n t/\hbar)}, \end{aligned} \quad (21)$$

$$e^{-iA_H t/\hbar} \psi_{|J_0, \gamma_0\rangle}(q) = \frac{1}{\sqrt{\mathcal{N}(J_0)}} \sum_n \sqrt{p_n(J_0)} e^{-i(\alpha_n \gamma_0 + E_n t/\hbar)} \psi_n(q). \quad (22)$$

Snapshots of the time evolution of the corresponding probability densities $|\sqrt{\mathcal{N}(J)} \langle J, \gamma | e^{-iA_H t/\hbar} | J_0, \gamma_0 \rangle|^2$ and $|e^{-iA_H t/\hbar} \psi_{|J_0, \gamma_0\rangle}(q)|^2$ are necessary in order to discriminate suitable functions $p_n(J)$ and sequences (α_n) from the classical limit viewpoint. Note that temporal evolution stability is granted with the choice $\alpha_n = \alpha_{-n} = E_n$.

V. THE QUEST FOR EXPLICIT PROBABILITIES $n \mapsto p_n(J)$: THE TWO SIMPLE CASES

The quantities that are left undetermined in our construction of the CS are the discretely indexed probability distribution $J \mapsto p_n(J)$ and the sequence $n \mapsto \alpha_n$. Two simplest situations are helpful in giving some hints: the free rotator (mass m on circle of radius l) and the harmonic oscillator (frequency ω), for which the energies are respectively $E_n \propto n^2 + \text{const.}$ and $E_n \propto n + \text{const.}$

In the first case, a familiar solution [6] is family of Gaussians centered at each integer, with dimensionless width parameter σ or $\epsilon \equiv 1/(2\sigma^2)$:

$$p_n(J) = \left(\frac{1}{2\pi\sigma^2} \right)^{1/2} e^{-\frac{1}{2\sigma^2 h^2} (J - hn)^2} \equiv \left(\frac{\epsilon}{\pi} \right)^{1/2} e^{-\epsilon(\tilde{J} - n)^2}, \quad n \in \mathbb{Z}, \quad (23)$$

with $\tilde{J} = J/h$. This gives the eigenvalues $J_n = hn$ of A_J and

$$E_n = \frac{h^2 n^2}{8\pi^2 m l^2} + \frac{\sigma^2 h^2}{8\pi^2 m l^2} = \frac{h^2 n^2}{8\pi^2 m l^2} + \frac{h^2}{16\epsilon \pi^2 m l^2}, \quad (24)$$

the constant shift being the average value of the classical energy with respect to the distribution $p_0(J)$. By introducing the Compton length $\lambda_c = \hbar/mc$ of the particle Eq. (24) reads $E_n =$

$(\lambda_c/l)^2 mc^2 (n^2 + 1/(2\epsilon))/2$. As an illustration of the suitability of this Gaussian choice, we show in Figure 1 the respective behaviors of the spectrum and lower symbol of the angle operator A_γ for different values of the parameter ϵ .

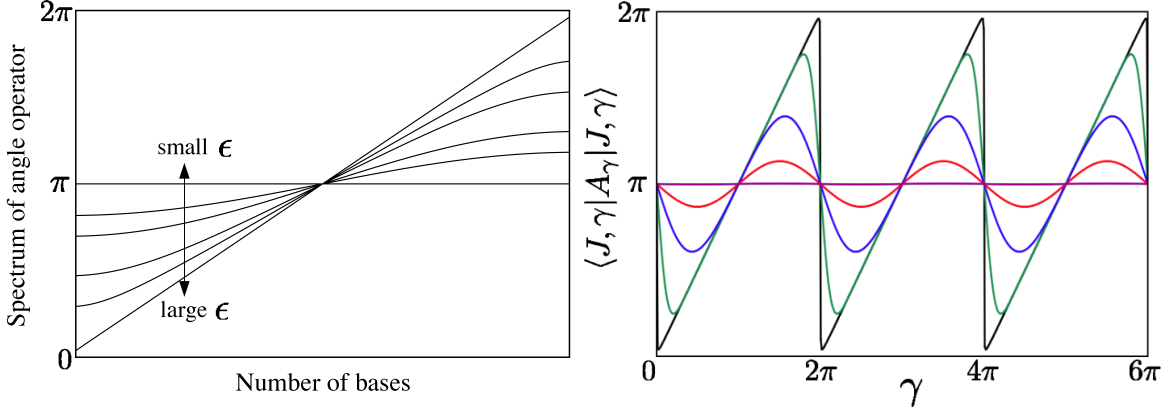


FIG. 1: Spectrum (left) and lower symbol (right) of the angle operator obtained by CS quantization of the angle of rotation on the circle, for different values of parameter $\epsilon = 1/(2\sigma^2)$, when $p_n(J) = (\frac{\epsilon}{\pi})^{1/2} e^{-\epsilon(\tilde{J}-n)^2}$ and $\alpha_n = n$. For the spectrum, $\epsilon = 10^{-7}, 0.3, 1, 3, 5, 50$. For the lower symbol: $\epsilon = 10^{-7}, 0.1, 1, 3, 10$ (the lower symbol has little dependence on J). One notices the tendency to the sawtoothed behavior of the classical angle function as $\epsilon \rightarrow \infty$, i.e. as $\sigma \rightarrow 0$.

In the second case, another familiar solution is the discretely indexed gamma distribution:

$$p_n(J) = e^{-\tilde{J}} \frac{\tilde{J}^n}{n!}, \quad n \in \mathbb{N}. \quad (25)$$

This gives $J_n = \hbar(n+1)$ and $E_n = \frac{\hbar\omega}{2\pi}(n+1) = \hbar\omega(n+1)$, the constant shift being the average value of the classical energy with respect to the distribution $p_0(J)$.

VI. THE QUEST FOR EXPLICIT PROBABILITIES: THE GENERAL CASE

Given the classical relation $E = E(J)$ between action variable and energy, and the observational or computed sequence (E_n) , the central question is to find the sequence of probability distributions $J \mapsto p_n(J)$ (at least with a satisfying approximation), which obey the two fundamental conditions (10). For the rotation case, the departure point could be a normal-like law, possibly modified along a perturbation scheme with expansion parameter the strength of the potential energy U . Similarly, in the libration case, the departure could be a gamma-like law, possibly

modified along the same lines. A nice and manageable model is the simple pendulum, whose the quantum version is well known from the solution of the Mathieu equation [8].

Approximations for the simple pendulum: the rotation case

Starting from the computed Mathieu eigenvalues E_n , an empirical approach consists in starting from the sequence of computed action variables $J_n^{\text{cl}} \stackrel{\text{def}}{=} J(E_n)$ from Eq. (1) and to impose, in the rotation case, the sequence of normal laws centered at these J_n^{cl} .

$$p_n(J) = \left(\frac{1}{2\pi\sigma_n^2} \right)^{1/2} e^{-\frac{1}{2\sigma_n^2 h^2} (J - J_n^{\text{cl}})^2}, \quad n \in \mathbb{Z}, \quad (26)$$

by “adjusting” σ_n in order to suitably approximate the E_n ’s with the computed quantities

$$E_n^{\text{app}} + \text{cst} \stackrel{\text{def}}{=} \int_{-\infty}^{+\infty} E(J) p_n(J) d\tilde{J}. \quad (27)$$

Note that with this choice, the eigenvalues J_n of A_J are precisely the J_n^{cl} ’s.

Approximations for the simple pendulum: the libration case

Handling the libration case is more delicate. Another empirical approach consists in introducing the gamma-like distribution:

$$J \mapsto p_n(J) = \frac{1}{\mathcal{E}_y(J)} \frac{\tilde{J}^n}{y_n!}, \quad n \in \mathbb{N}, \quad y_n! \stackrel{\text{def}}{=} y_1 y_2 \cdots y_n, \quad y_0! \stackrel{\text{def}}{=} 1, \quad \mathcal{E}_y(J) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \frac{\tilde{J}^n}{y_n!},$$

with an auxiliary sequence $\{0 = y_0 < y_1 < \cdots < y_n < \cdots\}$ such that the corresponding moment problem has a solution with a positive measure $w_y(J) d\tilde{J}$,

$$\int_0^{+\infty} d\tilde{J} w_y(J) p_n(J) = 1, \quad (28)$$

and that the quantization conditions involving the computed Mathieu eigenvalues,

$$E_n + \text{cst} = \int_0^{+\infty} d\tilde{J} w_y(J) E(J) p_n(J), \quad n = 0, 1, 2, \dots,$$

are fulfilled (at least approximately). We note that, by construction, we have $\sum_{n=0}^{\infty} p_n(J) = 1$, that the action variable J is the mean value of $n \mapsto y_n$ with respect to the Poisson-like distribution $n \mapsto p_n(J)$: $\langle y_n \rangle \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} y_n p_n(J) = J$, and that the quantum action A_J has eigenvalues $\langle J \rangle_J = y_{n+1}$.

VII. FREE ROTOR CS: TWO INTERESTING CHOICES

With the Gaussian choice (23) and for a general choice of a sequence of frequencies α_n , the coherent states for the free rotator read

$$|J, \gamma\rangle = \frac{1}{\sqrt{\mathcal{N}(J)}} \left(\frac{\epsilon}{\pi}\right)^{1/4} \sum_{n \in \mathbb{Z}} e^{-\frac{\epsilon}{2}(\tilde{J}-n)^2} e^{-i\alpha_n \gamma} |e_n\rangle, \quad (29)$$

The normalization function $\mathcal{N}(J)$ is given in two forms:

$$\mathcal{N}(J) = \sqrt{\frac{\epsilon}{\pi}} \sum_{n \in \mathbb{Z}} e^{-\epsilon(\tilde{J}-n)^2} \stackrel{\text{Poisson}}{=} \sum_{n \in \mathbb{Z}} e^{2\pi i n \tilde{J}} e^{-\frac{\pi^2}{\epsilon} n^2}, \quad (30)$$

and satisfies $\lim_{\epsilon \rightarrow 0} \mathcal{N}(J) = 1$. In the following we consider two choices of α and investigate what kind of properties of CS is satisfied for each case.

Case $\alpha_n = 2\pi n/\tau$

This choice renders exact the quantization of the classical canonical commutation rule $\{J, e^{i2\pi\gamma/\tau}\} = ie^{i2\pi\gamma/\tau}$. Indeed, we have $[A_J, A_{e^{i2\pi\gamma/\tau}}] = \hbar A_{e^{i2\pi\gamma/\tau}}$. Concerning the phase space distribution (7) for time evolution, we obtain the following upper bound:

$$\rho_{J_0, \gamma_0}(J, \gamma; t) \leq \frac{\epsilon}{\sqrt{\epsilon^2 + \tilde{t}^2}} \frac{e^{-\epsilon \frac{(\tilde{J} - \tilde{J}_0)^2}{2}}}{\mathcal{N}(J_0)} \sum_{n \in \mathbb{Z}} e^{-\frac{\epsilon}{2(\epsilon^2 + \tilde{t}^2)} (2\pi n - \tilde{\gamma} + \mu \tilde{t})^2}, \quad (31)$$

where $\epsilon = 1/(2\sigma^2)$, $\mu = (\tilde{J} + \tilde{J}_0)/2$, $\tilde{t} = \hbar t/(2ml^2)$, and $\tilde{\gamma} = 2\pi(\gamma - \gamma_0)/\tau$. From this is derived the estimate on the semi-classical behavior at large $\tilde{J}_0 = M \in \mathbb{Z}$: $\rho_{J_0, \gamma_0}(J, \gamma; t) \leq \frac{1}{\sqrt{1+4\sigma^4 \tilde{t}^2}} \delta_{\tilde{J} \tilde{J}_0}$ for $\tilde{\gamma}/2\pi - M\tilde{t} \in \mathbb{Z}$, and vanishes if $\tilde{J}_0 \notin \mathbb{Z}$ or $\tilde{\gamma}/2\pi - M\tilde{t} \notin \mathbb{Z}$.

Case $\alpha_n = 2\pi n^2/\tau$

This choice is appropriate to temporal evolution stability

$$e^{-iA_H t/\hbar} |J, \gamma\rangle = e^{-i\tilde{t}/2} \left| J, \gamma - \frac{\tau}{2\pi} \tilde{t} \right\rangle. \quad (32)$$

This time, the phase space distribution (7) is bounded as follows.

$$\lambda = \frac{2\pi}{\tau} (\gamma - \gamma_0 - \hbar t/(4\pi m l^2))$$

$$\rho_{J_0, \gamma_0}(J, \gamma; t) \leq \frac{\epsilon}{\sqrt{\epsilon^2 + \tilde{\gamma}^2(t)}} \frac{e^{-\epsilon \frac{(\tilde{J} - \tilde{J}_0)^2}{2}}}{\mathcal{N}(J_0)} \sum_{n \in \mathbb{Z}} e^{-\frac{\epsilon}{2(\epsilon^2 + \tilde{\gamma}^2(t))} (2\pi n - 2\mu \tilde{\gamma}(t))^2}, \quad (33)$$

where $\tilde{\gamma}(t) = \frac{2\pi}{\tau}(\gamma - \gamma_0 - \hbar t/(4\pi m l^2)) = \tilde{\gamma} - \tilde{t}$. Hence the estimate on semi-classical behavior of $\rho_{J_0, \gamma_0}(J, \gamma; t)$ at large $\tilde{J}_0 = M \in \mathbb{Z}$: $\rho_{J_0, \gamma_0}(J, \gamma; t) \leq \frac{1}{\sqrt{1+4\sigma^4\tilde{\gamma}^2(t)}} \delta_{\tilde{J}\tilde{J}_0}$ for $\tilde{\gamma}(t) \in 2\pi\mathbb{Z}/M$, and vanishes if $\tilde{J}_0 \notin \mathbb{Z}$ or $\tilde{\gamma}(t) \notin 2\pi\mathbb{Z}/M$.

VIII. CONCLUDING POINTS

Periodic and more generally Integrable systems provide a variety of such families of action-angle coherent states. A future issue is about the best choice of probability distributions $n_i \mapsto p_{n_i}(J_i)$. A more fundamental concerns the physical (in terms of physical measurement) equivalence between different families of coherent states from a quantization point of view. Finally, the extension of the probabilistic approach presented in this work to unconfined systems and subsequent continuous spectra is possible [7].

-
- [1] Goldstein H, Poole C, and Safko J 1981 Classical Mechanics 3rd ed Addison Wesley
 - [2] Gazeau J P Heller B, and Kanamoto R Action-angle coherent states and related quantization (in preparation)
 - [3] Gazeau J P 2009 Coherent States in Quantum Physics Wiley-VCH
 - [4] Gazeau J P and Klauder J R 1999 Coherent States for Systems with Discrete and Continuous Spectrum *J. Phys. A: Math. Gen.* **32** 123
 - [5] Ali A H, Heller B and Gazeau J P 2008 Coherent states and Bayesian duality, *J. Phys. A: Math. Theor.* **41** 365302
 - [6] Kowalski K, Rembieliński J, and Papaloucas L C 1996 Coherent states for a quantum particle on a circle, *J. Phys. A: Math. Gen.* **29** 4149
 - [7] Bagrov V G, Gazeau J P, Gitman D, and Levine A Coherent states and related quantizations for unbounded motions (in preparation)
 - [8] Abramowitz M and Stegun I A eds. 1972 Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables Dover Publications